DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA

FIRST SEMESTER EXAMINATION 2024/2025
MATH 307: COMPLEX ANALYSIS I

DATE: April, 2025 TIME: 2 HOURS

INSTRUCTIONS: ATTEMPT ANY FOUR QUESTIONS

1. (a) If z is a complex variable, prove the following identities:

   (i) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (ii) $\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$

   (b) Prove that a single valued complex function $f(z)$ is continuous at a point z_0 if and only if $\lim_{z\to z_0} f(z) = f(z_0)$

2. (a) Evaluate the following limits (i) $\lim_{z\to -2i}\frac{(z-5)(z+7)}{2z-i}$ (ii) $\lim_{z\to 4i}\frac{z^2+81}{z-9i}$ (iii) $\lim_{n\to\infty}\frac{2n^2-3in+3}{2n^2+in-2}$

   (b) Investigate the continuity of $f(z)=\begin{cases}\dfrac{3z^4-2z^3+8z^2-2z+5}{z-i} & for\ z\neq i\\ 2+3i & for\ z=i\end{cases}$ at $z = i$.

   (c) What happens in the case of question 2(b) above if the value of $f(z)$ at $z = i$ is $4 + 4i$

3. (a) Prove that if $f(z)$ is continuous in a closed and bounded region, then it is bounded in the region.

   (b) (i) Prove that $\lim_{n\to\infty}\frac{2n-7i}{2n+5i}=1$ (ii) investigate the convergence of $\sum_{n=0}^{\infty}\frac{(z-i)^n}{(2+i)^{n+1}}$

   (iii) Distinguish between absolute and conditional convergence of series

4. (a) Prove that if $\sum_{n=1}^{\infty} z_n$ converges to S_1 and $\sum_{n=1}^{\infty} w_n$ converges to S_2, then $\sum_{n=1}^{\infty}(z_n - w_n)$ converges to $S_1 - S_2$

   (b) Prove that the series $1 + z + z^2 + z^3 + \ldots\ldots.$ converges to $\frac{1}{1-z}$ *if* $|z|<1$. Is the convergence uniform? Prove your assertion.

   (c) Evaluate $\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)$

5. (a) Derive Cauchy Riemann equations in Cartesian form.

   (b) Investigate the differentiability of the following complex functions

   (i) $f(z)=x^2+iy^2-7$ (ii) $f(z)=y^2-ix^2+5$ (iii) $f(z)=|z|^2+9$

6. (a) Let $f(z)$ be analytic everywhere on and within a closed contour C, positively oriented. If z_0 is a point within C, prove that $f(z_0)=\int_C \frac{f(z)}{z-z_0}dz$

   (b) Use the result in 6(a) or otherwise to evaluate the following integral

   $$\oint_C \frac{ze^{3z}+1}{(z-5)(z^2-1)}dz \qquad \text{where } C\text{: rectangle } -3\le x\le 3;\ \frac{-5}{2}\le y\le\frac{5}{2}$$

(1a i) $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

from RHS,

$$\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \left[\frac{e^{iz_1}-e^{-iz_1}}{2i}\right]\left[\frac{e^{iz_2}+e^{-iz_2}}{2}\right] + \left[\frac{e^{iz_1}+e^{-iz_1}}{2}\right]\left[\frac{e^{iz_2}-e^{-iz_2}}{2i}\right]$$

$$= \left[\frac{e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}}{4i}\right] + \left[\frac{e^{i(z_1+z_2)}-e^{i(z_1-z_2)}+e^{i(z_2-z_1)}-e^{-i(z_1+z_2)}}{4i}\right]$$

$$= \frac{2e^{i(z_1+z_2)}-2e^{-i(z_1+z_2)}}{4i} = \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i} = \sin(z_1+z_2)$$

Hence;

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2. \quad \square$$

(ii) $\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$

Let $w = \tanh^{-1} z \Rightarrow z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^w - e^{-w}}{e^w + e^{-w}}$

$$\Rightarrow z(e^w + e^{-w}) = e^w - e^{-w} \Rightarrow ze^w + ze^{-w} = e^w - e^{-w}$$

$$ze^w - e^w = -e^{-w} - ze^{-w} \Rightarrow e^w(z-1) = -e^{-w}(1+z)$$

$$\Rightarrow \frac{e^w\cancel{(1-z)}}{e^{w}\cancel{(1-z)}} = \frac{e^{-w}(1+z)}{e^{-w}(1-z)} \Rightarrow \frac{1+z}{1-z} = e^{2w}$$

$\Rightarrow$ for a complete solution, $e^{2w} = e^{2(w-k\pi i)}$ for $k = 0, \pm1, \pm2, \cdots$

Taking log of both sides, we have;

$$\ln\left(\frac{1+z}{1-z}\right) = 2w \Rightarrow \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = w$$

$$\Rightarrow w - k\pi i = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \text{ for } k = 0, \pm1, \pm2, \cdots$$

choosing the principal branch of $\tanh z$, $k=0$, then;

$$w = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow \tanh^{-1} z = w = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \quad \square$$

(1b) Proof:

Since $f(z)$ is continuous at a point z_0, then for any $\varepsilon > 0 \; \exists \; |f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

In particular, $|f(z) - f(z_0)| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

$\Rightarrow \lim_{z \to z_0} f(z) = f(z_0)$.

Conversely, Suppose $\lim_{z \to z_0} f(z) = f(z_0)$. Then for any $\varepsilon > 0 \; \exists \; \delta(\varepsilon) > 0 \; \exists$ $|f(z) - f(z_0)| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

However, at $z = z_0$, $|f(z) - f(z_0)| = |f(z_0) - f(z_0)| = |0| = 0$.

Now, for the interval, $|z - z_0|$, we have $|f(z) - f(z_0)| < \varepsilon$.

Thus, $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

$\Rightarrow f(z)$ is continuous at the point z_0. $\square$.

(2a)
$$\lim_{z \to 2i} \frac{(z-5)(z+7)}{2z - i} = \frac{(2i-5)(2i+7)}{2(2i) - i} = \frac{-4 + 14i - 10i - 35}{4i - i} = \frac{4i - 39}{3i} = \frac{4}{3} - \frac{13}{i}$$

$$= \frac{-13i}{-1} + \frac{4}{3} = \underline{\underline{13i + \frac{4}{3}}}$$

OR

$$\lim_{z \to 2i} \frac{(z-5)(z+7)}{2z - i} = \lim_{z \to 2i} \frac{(z^2 + 2z - 35)}{2z - i} = \frac{(2i)^2 + 2(2i) - 35}{2(2i) - i} = \frac{-4 + 4i - 35}{4i - i}$$

$$= \frac{4i - 39}{3i} = \frac{4}{3} - \frac{39}{3i} = \frac{4}{3} - \frac{13}{i} = \underline{\underline{13i + \frac{4}{3}}}$$

(ii)
$$\lim_{z \to 4i} \frac{z^2 + 81}{z - 9i} = \lim_{z \to 4i} \frac{\cancel{(z - 9i)}(z + 9i)}{\cancel{(z - 9i)}} = \lim_{z \to 4i} z + 9i = 4i + 9i = \underline{\underline{13i}}$$

(iii)
$$\lim_{n \to \infty} \frac{2n^2 - 3in + 3}{2n^2 + in - 2} = \lim_{n \to \infty} \left[\frac{\left(\frac{2n^2}{n^2} - \frac{3in}{n^2} + \frac{3}{n^2}\right)}{\left(\frac{2n^2}{n^2} + \frac{in}{n^2} - \frac{2}{n^2}\right)} \right] = \lim_{n \to \infty} \frac{\left(2 - \cancel{\frac{3i}{n}}^{0} + \cancel{\frac{3}{n^2}}^{0}\right)}{2 + \cancel{\frac{i}{n}}^{0} - \cancel{\frac{2}{n^2}}^{0}}$$

$$= \frac{2}{2} = \underline{\underline{1}}$$

(2b) To investigate the continuity of a complex function,

(i) $f(z)$ must exist at z_0,

(ii) $\lim_{z \to z_0} f(z)$ must exist and

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$.

$$f(z) = \begin{cases} \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} & \text{for } z \neq i \\ 2 + 3i & \text{for } z = i \end{cases} \quad \text{at } z = i$$

$z_0 = i$ Now, $f(i) = 2 + 3i$

$$\lim_{z \to z_0} f(z) = \lim_{z \to i} f(z) = \lim_{z \to i} \left(\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} \right)$$

$$= \lim_{z \to i} \frac{(3z^3 - (2-3i)z^2 + (5-2i)z + 5i)\cancel{(z-i)}}{\cancel{(z-i)}} = \lim_{z \to i} 3z^3 - (2-3i)z^2 + (5-2i)z + 5i$$

$$= 3(i)^3 - (2-3i)(i)^2 + (5-2i)i + 5i = \cancel{-3i + 2 - 3i + 5i + 2 + 5i} =$$

$$= -3i + 2 - 3i + 5i + 2 + 5i = 4i + 4$$

Clearly, we can see that $\lim_{z \to z_0} f(z) \neq f(z_0)$.

Hence, the function is not cont. at $z = i$

(2c) If the value of $f(z)$ at $z = i = 4 + 4i$

Then, we can conclude that the function is cont. at that point $z = i$

Because $\lim_{z \to z_0} f(z) = f(z_0)$ D.

(3a) Proof:

Since $f(z)$ is cont. on R, then for any $z_0 \in R$ and $\varepsilon > 0$ $\exists$ a $\delta_{z_0}(\varepsilon) > 0$ $\ni$ $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta_{z_0}$. In particular, if $\varepsilon = 1$, then $|f(z) - f(z_0)| < 1$ whenever $|z - z_0| < \delta_{z_0}$.

$|f(z)| - |f(z_0)| \leq |f(z) - f(z_0)| < 1 \Rightarrow |f(z)| < 1 + |f(z_0)|$ whenever $|z - z_0| < \delta_{z_0}$

The sets $|z - z_0|$ are infinite in the region R where they cover whole region. However since R is closed and bdd, then by Heine-Borel theorem, the region is compact i.e for any open covering of R $\exists$ a finit sub-cover.

This implies $\exists$ a finite no $|z - z_0|$ which cover R. If this finite no is K say, then $|f(z)| < [1 + f(z_0)]_1, [1 + f(z_0)]_2, \dots, [1 + f(z_0)]_K$

If the maximum value is M

$\Rightarrow f(z) < \max [1+f(z_0)]_1, [1+f(z_0)]_2, \ldots, [1+f(z_0)]_k$.

If the maximum value is M then $f(z) < M$ $\forall$ point in R

$\Rightarrow f(z)$ is bdd in R. $\square$.

3bi) $\lim_{n\to\infty} \frac{2n-7i}{2n+5i} = 1$

We need to proof that for any $\varepsilon > 0$ $\exists$ a $\delta(\varepsilon) > 0$ $\ni$ $|f(z)-1| < \varepsilon$ whenever $0 < |z-1| < \varepsilon$.

$\Rightarrow \left|\frac{2n-7i}{2n+5i} - 1\right| < \varepsilon$ whenever $0 < |z-1| < \varepsilon$.

$\Rightarrow \left|\frac{2n-7i-2n-5i}{2n+5i}\right| = \left|\frac{-12i}{2n+5i}\right| = \frac{12}{|2n+5i|} \leq \frac{12}{|2n|-|5i|} \leq \frac{12}{2n-10}$

$= \frac{6}{n-5}$ So that if we choose $\varepsilon > 0$ then $\frac{6}{n-5} < \varepsilon$ provided

$n-5 > 6/\varepsilon$ or $n > \frac{6}{\varepsilon} + 5$.

Therefore, $|z_n - 1| = \frac{6}{n-5} < \varepsilon$ $\forall n \geq N$.

$\therefore \lim_{n\to\infty} \frac{2n-7i}{2n+5i} = 1$ $\square$.

(ii) To investigate the convergence of a series, we use some methods such as the ratio test.

$\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2+i)^{n+1}}$. Using the ratio test, we have

$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty} \left|\frac{(z-i)^{n+1}}{(2+i)^{n+2}} \cdot \frac{(2+i)^{n+1}}{(z-i)^n}\right| = \lim_{n\to\infty} \left|\frac{z-i}{2+i}\right| = \frac{|z-i|}{|2+i|} < 1$

provided $|z-i| < |2+i|$ or $|z-i| < \sqrt{5}$

The series converges absolutely for all values in the circle $|z-i| < \sqrt{5}$ //

3iii) A series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges while if a series $\sum_{n=1}^{\infty} z_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty} z_n$ converges but $\sum_{n=1}^{\infty} |z_n|$ does not converge.

(4a) Since $\sum_{n=1}^{\infty} Z_n$ converges to S_1 and $\sum_{n=1}^{\infty} k_n$ converges to S_2 then the sequences of their partial sum $\{S_n\}$ and $\{K_n\}$ converges to A_1 and A_2 respectively.

Then, for any $\varepsilon > 0$ $\exists N_1(\varepsilon), N_2(\varepsilon) \ni |S_n - A_1| < \varepsilon/2 \ \forall n \geq N_1$ and $|K_n - A_2| < \varepsilon/2 \ \forall n \geq N_2$.

$\therefore |(S_n - K_n) - (A_1 - A_2)| = |S_n - A_1 - K_n + A_2| \leq |S_n - A_1| + |-(K_n - A_2)| < \varepsilon/2 + \varepsilon/2$

$= \varepsilon \ \forall n \geq N_1, N_2$.

If we choose $N = \mathrm{Max}(N_1, N_2)$ then;

$|(S_n - K_n) - (A_1 - A_2)| < \varepsilon \ \forall n \geq N$.

Since the sequence of their partial sums $S_n - K_n$ converges to $A_1 - A_2$ then the series $\sum_{n=1}^{\infty}(Z_n - k_n)$ converges to $(S_1 - S_2)$ also. $\square$

(4b) Consider $S_n = 1 + z + z^2 + \dots + z^n$ and $zS_n = z + z^2 + \dots + z^{n-1} + z^n$

Substracting the two equations gives

$S_n - zS_n = 1 - z^n$

$S_n(1-z) = 1 - z^n \Rightarrow S_n = \dfrac{1-z^n}{1-z}$. So that,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1-z^n}{1-z}\right) = \frac{1 - \lim_{n\to\infty} z^n}{1-z}$$

But $\lim_{n\to\infty} z^n$, $z, z^2, z^3, z^4, \dots$

Since $|z| < 1$ then $1/2 < 1$

$\Rightarrow \lim_{n\to\infty} z^n = 0$

Hence, $\lim_{n\to\infty}\left(\dfrac{1-z^n}{1-z}\right) = \dfrac{1-0}{1-z} = \dfrac{1}{1-z}$

$$\therefore \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z} \quad \square$$

The convergence is not Uniform or not Uniformly Convergence.

Proof:

The sequence of partial sum $\{S_n\} = \left\{\dfrac{1-z^n}{1-z}\right\} \Rightarrow \lim_{n\to\infty}\left\{\dfrac{1-z^n}{1-z}\right\} = \dfrac{1}{1-z}$ for $|z| < 1$

$\Rightarrow \left|\dfrac{1-z^n}{1-z} - \dfrac{1}{1-z}\right| = \left|\dfrac{z^n}{1-z}\right| = \dfrac{|z|^n}{1-z} < \varepsilon$ provided $|z^n| < \varepsilon(1-z)$ or

$\dfrac{n\log|z|}{\log|z|} < \dfrac{\log \varepsilon(1-z)}{\log|z|}$ or $n > \dfrac{\varepsilon(1-z)}{|z|}$ since $|z| < 1$

If we chooce N to be the smallest positive integer greater than $\dfrac{\varepsilon(1-z)}{|z|}$ and in that case;

$$\left|\frac{z^n}{1-z}\right| \leq \left|\frac{z^N}{1-z}\right| < \varepsilon \ \forall n \geq N$$

In this case, N is dependent on both ε and z and so the series is not uniformly convergence.

(4c) $\lim_{n\to\infty}(\sqrt{n}-\sqrt{n+1}) = \lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\cdot\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}\right) = \lim_{n\to\infty}\left(\frac{n-n-1}{\sqrt{n}+\sqrt{n+1}}\right)$

$= \lim_{n\to\infty}\left(\frac{-1}{\sqrt{n}+\sqrt{n+1}}\right) = \frac{-1}{\sqrt{\infty}+\sqrt{\infty+1}} = \frac{-1}{\infty+\infty} = \frac{1}{\infty} = \underline{\underline{0}}$

(5a) Let $f(z)=u(x,y)+iv(x,y)$ and $f(z)$ is differentiable at z_0 where $z_0=x_0+iy_0$.
If we let $\Delta z=\Delta x+i\Delta y$ then $\Delta u = u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)$ and
$\Delta v = v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)$.
If $f'(z_0)=a+ib \Rightarrow \lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = a+ib \Rightarrow \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y} = a+ib$

$\Rightarrow \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \mathrm{Re}\left\{\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y}\right\} = a$ and $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \mathrm{Im}\left\{\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y}\right\} = b$

Suppose $\Delta y\to 0$ then,

$\lim_{\Delta x\to 0}\frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} = a$ and $\lim_{\Delta x\to 0}\frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x} = b$ —— (*)

$\Rightarrow \frac{\partial u}{\partial x} = a$ and $\frac{\partial v}{\partial x} = b$ —— (1)

On the other hand, if $\Delta x\to 0$ then;

$\lim_{\Delta y\to 0}\frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{\Delta y} = a$ and $\lim_{\Delta y\to 0}\frac{-[u(x_0, y_0+\Delta y)-u(x_0,y_0)]}{\Delta y} = b$ —— (*)

$\Rightarrow \frac{\partial v}{\partial y} = a$ and $-\frac{\partial u}{\partial y} = b$ —— (2)

$\Rightarrow \frac{\partial u}{\partial x} = a = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and

$\frac{\partial v}{\partial x} = b = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

From the values of $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}$, we have the C-R equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ //

(5b) i) $f(z) = x^2+iy^2-7 \Rightarrow u = x^2-7$ and $v=y^2$.
$\Rightarrow \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 2y$.
C-R equation; $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x = 2y \Rightarrow x=y$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 0=0$
$\Rightarrow$ The C-R equation is not satisfied. Hence, it is not differentiable.

5ii) $f(z) = y^2 - ix^2 + 5 \Rightarrow u = y^2+5$ and $v = -x^2$

$\Rightarrow \frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial v}{\partial x} = -2x$ and $\frac{\partial v}{\partial y} = 0$ which all exists and are continuous.

Now we check the C-R equation;

$\Rightarrow$ C-R eqn $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 0 = 0$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow -2x = -2y \Rightarrow x = y$.

Since the C-R equation is not satisfied then we can conclude that the function is not differentiable.

5iii) $f(z) = |z|^2 + 9 = |x+iy|^2 + 9 = (\sqrt{x^2+y^2})^2 + 9 = x^2+y^2+9$.

$\Rightarrow u = x^2+y^2+9$ and $v = 0 \Rightarrow \frac{\partial u}{\partial x} = 2x,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial u}{\partial y} = 2y$ and $\frac{\partial v}{\partial y} = 0$,

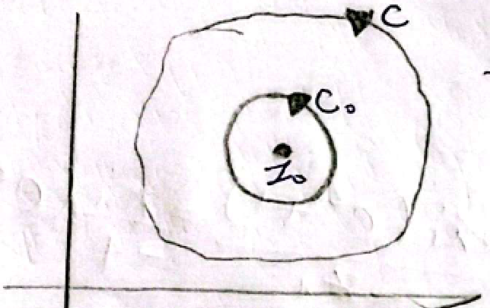
The C-R equation, $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x = 0 \Rightarrow x = 0$

and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 0 = -2y \Rightarrow y = 0$.

Hence, it does not satisfy the C-R equation. But it is true only if $x=0, y=0$ or at $(x,y) = (0,0)$. Therefore, the C-R equation are only satisfied at the point $(0,0)$ implying that the function is differentiable at $(0,0)$. Then the derivative of the function will then be;

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(0) = \underline{2x}.$$

6a) Proof:



Let C_0 be the circle $|z-z_0| \le r_0$ where r_0 is chosen in such a way that C_0 is completely within C. The integral $\frac{f(z)}{z-z_0}$ is analytic in the region bounded by C and C_0 as well as on C_0. By C.G theorem,

$$\int_C \frac{f(z)}{z-z_0}dz = \int_{C_0} \frac{f(z) - f(z_0) + f(z_0)}{z-z_0}dz = \int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz + \int_{C_0}\frac{f(z_0)}{z-z_0}dz$$

$$= f(z_0)\int_{C_0}\frac{dz}{z-z_0} + \int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz.$$

However, On C_0, $|z-z_0| = r_0 \Rightarrow z - z_0 = r_0 e^{i\theta}$, $0 \le \theta \le 2\pi$ and so,

$z = z_0 + r_0e^{i\theta} \Rightarrow dz = ir_0e^{i\theta}d\theta \Rightarrow \int_0^{2\pi}\frac{dz}{z-z_0} = \int_0^{2\pi}\frac{ir_0e^{i\theta}d\theta}{z_0+r_0e^{i\theta}-z_0} = \int_0^{2\pi} i\,d\theta$

$= i\theta\Big|_0^{2\pi} = 2\pi i$

$\Rightarrow \int_{C_0}\frac{f(z_0)}{z-z_0}dz = 2\pi i f(z_0) \Rightarrow \int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) + \int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz$

But $\int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}dz = 0$

$\Rightarrow \int_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0) \Rightarrow f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$ □.

(6b) $\oint_C \frac{ze^{3z}+1}{(z-5)(z^2-1)}\,dz$ where C: rectangle $-3 \le x \le 3;\ -\frac{5}{2} \le y \le 5/2$

On C_1, let $f(z) = \frac{ze^{3z}+1}{(z-5)(z-1)}$ and $z_0 = -1$.

also, on C_2, $f(z) = \frac{ze^{3z}+1}{(z-5)(z+1)}$ and $z_0 = 1$

$$\Rightarrow \oint_C \frac{ze^{3z}+1}{(z-5)(z^2-1)}\,dz = \oint \frac{ze^{3z}+1}{(z-5)(z-1)(z+1)}\,dz = \int_{C_1} \frac{\frac{ze^{3z}+1}{(z-5)(z-1)}}{z+1}\,dz + \int_{C_2} \frac{\frac{ze^{3z}+1}{(z-5)(z+1)}}{z-1}\,dz$$

$\Rightarrow$ Now Using C-I formula

$$\Rightarrow \int_{C_1} \frac{ze^{3z}+1}{(z-5)(z+1)}\,dz \text{ at } z_0 = -1 = 2\pi i\left(\frac{-1e^{-3}+1}{-6\times-2}\right) = \cancel{2}\pi i\left(\frac{-e^{-3}+1}{\cancel{12}}\right)$$

$$= \pi i\left(\frac{-e^{-3}+1}{6}\right) = \frac{\pi i(1-e^{-3})}{6}$$

Also,

$$\int_{C_2} \frac{ze^{3z}+1}{(z-5)(z+1)}\,dz \text{ at } z_0 = 1$$

$$= 2\pi i f(1) = 2\pi i\left(\frac{e^3+1}{-4\times 2}\right) = \frac{\cancel{2}\pi i\,(e^3+1)}{-\cancel{8}\,4} = \frac{\pi i}{-4}(e^3+1)$$

$$\text{Hence } \oint_C \frac{ze^{3z}+1}{(z-5)(z^2-1)}\,dz = \frac{\pi i}{6}(1-e^{-3}) - \frac{\pi i}{4}(e^3+1)$$

$$= \pi i\left[\left(\frac{1-e^{-3}}{6}\right) - \left(\frac{e^3+1}{4}\right)\right] = \pi i\left[\frac{2-2e^{-3}-3e^3-3}{12}\right]$$

$$= \frac{-\pi i}{12}\left[2e^{-3}+3e^3+1\right] = \underline{\frac{-\pi i}{12}(2e^{-3}+3e^3+1)}$$

$$= \cancel{\frac{-\pi i}{12}(2+3e^6+e}$$